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## Relativistic Hydrodynamic Codes for Adiabatic and Isothermal Flows

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### Abstract.

The equation of state (EOS) is an important issue in numerical simulation codes for relativistic hydrodynamics. We describe a code for adiabatic flows, employing an EOS which is simple and yet approximates very closely the EOS of perfect gas in relativistic regime. We also describe a code for isothermal flows, where the EoS is trivially given.

### 1. Introduction

Highly energetic phenomena, which are relativistic in nature, are common in astrophysical environments: accretion disks around black holes (see, e.g., Miller 2007, for review), relativistic jets from Galactic sources (see, e.g., Mirabel & Rodríguez 1999, for review), extragalactic jets from active galactic nuclei (see, e.g., Zensus 1997, for review), and gamma-ray bursts (see, e.g., Mészáros 2002, for review). Gas in such relativistic phenomena is characterized by its relativistic fluid speed ( $v \rightarrow c$ ) and/or relativistic sound speed ( $c_s \rightarrow c/\sqrt{3}$ ).

Numerical codes for relativistic hydrodynamics (RHDs) have been successfully built, based on schemes that were originally developed for codes for non-relativistic hydrodynamics. Most codes employed the equation of state (EoS) of the gas, which was designed for the gas with a constant ratio of specific heats and so is essentially valid only for the gas in either non-relativistic or ultra-relativistic regime (see, e.g., Martí & Müller 2003; Wilson & Mathews 2003, for reviews). The correct EoS (see Section 3), however, involves the specific enthalpy expressed in terms of the modified Bessel functions (see Synge 1957). While codes employing the correct EoS has been built (see, e.g., Falle & Komissarov 1996; Scheck et al. 2002), they normally comes with an extra cost of computation time. On the other hand, approximate EoSs that mimic the correct EoS have been suggested (see, e.g., Mathews 1971; Service 1986; Ryu et al. 2006), and recently, codes employing those approximate EoSs have been introduced (see, e.g., Mignone et al. 2005; Ryu et al. 2006).

In this paper, we describe a RHD code for adiabatic flows, which was presented in Ryu et al. (2006); an EoS, which is simple and an algebraic function of temperature, was employed. We also describe a new RHD code for isothermal flows; the meaning of isothermality in relativistic regime is discussed. The steps necessary to build codes including the transformation from the conserved quantities to the primitive quantities

and the eigen-structure are presented. Finally, shock tube tests performed with codes based on the Total Variation Diminishing (TVD) scheme are presented.

## 2. Relativistic Hydrodynamic Equations

The special RHD equations for an ideal fluid in the laboratory frame of reference can be written as

$$\frac{\partial D}{\partial t} + \frac{\partial}{\partial x_j} (D v_j) = 0, \quad (1)$$

$$\frac{\partial M_i}{\partial t} + \frac{\partial}{\partial x_j} (M_i v_j + p \delta_{ij}) = 0, \quad (2)$$

$$\frac{\partial E}{\partial t} + \frac{\partial}{\partial x_j} [(E + p) v_j] = 0, \quad (3)$$

where  $D$ ,  $M_i$ , and  $E$  are the mass density, momentum density, and total energy density, respectively (see, e.g., Landau & Lifshitz 1959). The conserved quantities in the laboratory frame are expressed as

$$D = \Gamma \rho, \quad M_i = \Gamma^2 \rho h v_i, \quad E = \Gamma^2 \rho h - p, \quad (4)$$

where  $\rho$ ,  $v_i$ ,  $p$ , and  $h$  are the proper mass density, fluid three-velocity, isotropic gas pressure and specific enthalpy, respectively, and the Lorentz factor is given by

$$\Gamma = \frac{1}{\sqrt{1 - v^2}} \quad \text{with} \quad v^2 = v_x^2 + v_y^2 + v_z^2. \quad (5)$$

Here,  $h \equiv (e + p)/\rho$ , where  $e$  is the sum of the proper internal and rest-mass energy densities. In the above, the Latin indices (e.g.,  $i$ ) represents spatial coordinates and the conventional Einstein summation is used. The speed of light is set to unity ( $c \equiv 1$ ) throughout this paper.

## 3. Code for Adiabatic Flows

Equations (1) – (3) with the EoS,  $h = h(p, \rho)$ , form a hyperbolic set of conservation equations for adiabatic flows. The correct EoS for the single-component perfect gas in relativistic regime (hereafter RP) can be derived (see Synge 1957); it is given as

$$h = \frac{K_3(1/\Theta)}{K_2(1/\Theta)}, \quad (6)$$

where  $K_2$  and  $K_3$  are the modified Bessel functions of the second kind of order two and three, respectively. Here,  $\Theta = p/\rho$  is a temperature-like variable. Using the EoS in (6), however, poses a difficulty, because the inverse, that is,  $\Theta$  as a function of  $h$ , can not be expressed as a simple form.

Ryu et al. (2006) introduced an approximate EoS (hereafter RC),

$$h = 2 \frac{6\Theta^2 + 4\Theta + 1}{3\Theta + 2}. \quad (7)$$

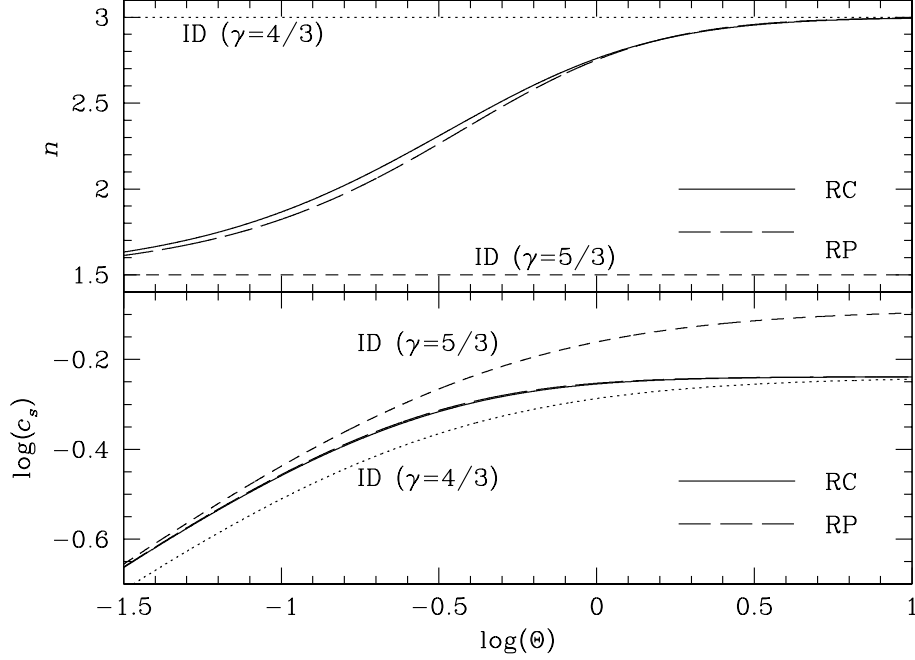


Figure 1. Polytropic index,  $n$ , and sound speed,  $c_s$ , as a function of temperature,  $\Theta = p/\rho$ , for adiabatic flows employing different EoSs.

RC mimics very closely RP. Figure 1 compares the polytropic index and sound speed

$$n = \rho \frac{\partial h}{\partial p} - 1, \quad c_s^2 = -\frac{\rho}{nh} \frac{\partial h}{\partial \rho}, \quad (8)$$

for RP and RC as well as for the EoS with a constant ratio of specific heats,  $\gamma$ , (hereafter ID, standing for ideal gas). The specific enthalpy  $h$  for RC fits  $h$  for RP within the error of 0.8 %.

Building codes based on upwind schemes requires the eigen-structure (eigenvalues and eigenvectors) for the relevant hyperbolic set of conservation equations. The eigen-structure for RHD equations for adiabatic flows in (1) – (3) are given in Ryu et al. (2006). The eigenvalues are

$$a_1 = \frac{(1 - c_s^2)v_x - c_s/\Gamma \cdot \sqrt{Q}}{1 - c_s^2 v^2}, \quad (9)$$

$$a_2 = a_3 = a_4 = v_x, \quad (10)$$

$$a_5 = \frac{(1 - c_s^2)v_x + c_s/\Gamma \cdot \sqrt{Q}}{1 - c_s^2 v^2}, \quad (11)$$

where  $Q = 1 - v_x^2 - c_s^2(v_y^2 + v_z^2)$ , assuming that the flow varies along the  $x$ -direction. For the right and left eigenvectors, refer Ryu et al. (2006).

Equations (1) – (3) evolve the conserved quantities  $D$ ,  $M_i$ , and  $E$ , but the primitive quantities,  $\rho$ ,  $v_i$ , and  $p$ , are necessary to calculate the eigenvalues and eigenvectors.

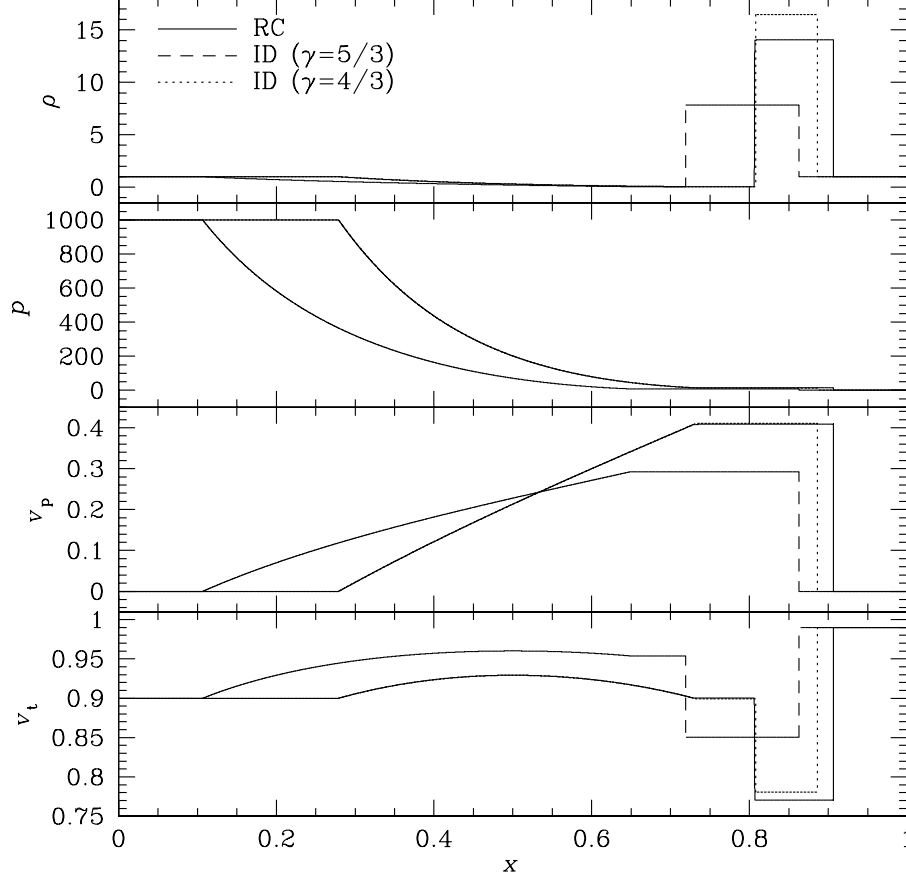


Figure 2. A relativistic shock tube from the code for adiabatic flows employing different EoSs.

Combining (4) – (5) along with (7) results in

$$\begin{aligned}
 & M \sqrt{\Gamma^2 - 1} \left[ 3E\Gamma(8\Gamma^2 - 1) + 2D(1 - 4\Gamma^2) \right] \\
 & = 3\Gamma^2 \left[ 4(M^2 + E^2)\Gamma^2 - (M^2 + 4E^2) \right] - 2D(4E\Gamma - D)(\Gamma^2 - 1). \quad (12)
 \end{aligned}$$

Further simplification reduces this to an equation involving the 8<sup>th</sup> power of  $\Gamma$ . The above can be solved numerically to get  $v$ . And then, the rest of the primitive quantities can be solved.

A RHD code for adiabatic flows was built based on the TVD scheme (Ryu et al. 2006). Figure 2 show a shock tube test comparing the results with RC and ID: initially  $\rho_L = \rho_R = 1$ ,  $p_L = 10^3$ ,  $p_R = 10^{-2}$ ,  $v_{p,L} = v_{p,R} = 0$ ,  $v_{t,L} = 0.9$ ,  $v_{t,R} = 0.99$ , and  $t_{\text{end}} = 0.75$ . Here, the subscripts  $L$  and  $R$  denote the quantities in the left and right states of the initial discontinuity at  $x = 0.5$ , and  $t_{\text{end}}$  is the time when the solutions are presented. And  $v_p$  and  $v_t$  are the velocity components parallel and transverse to the propagation of structures (i.e., the  $x$ -direction). The ID solution with  $\gamma = 4/3$  matches well to the RC solution in the left of the contact discontinuity where the flow has  $\Theta \gg 1$ . But a difference is obvious in the region between contact discontinuity and

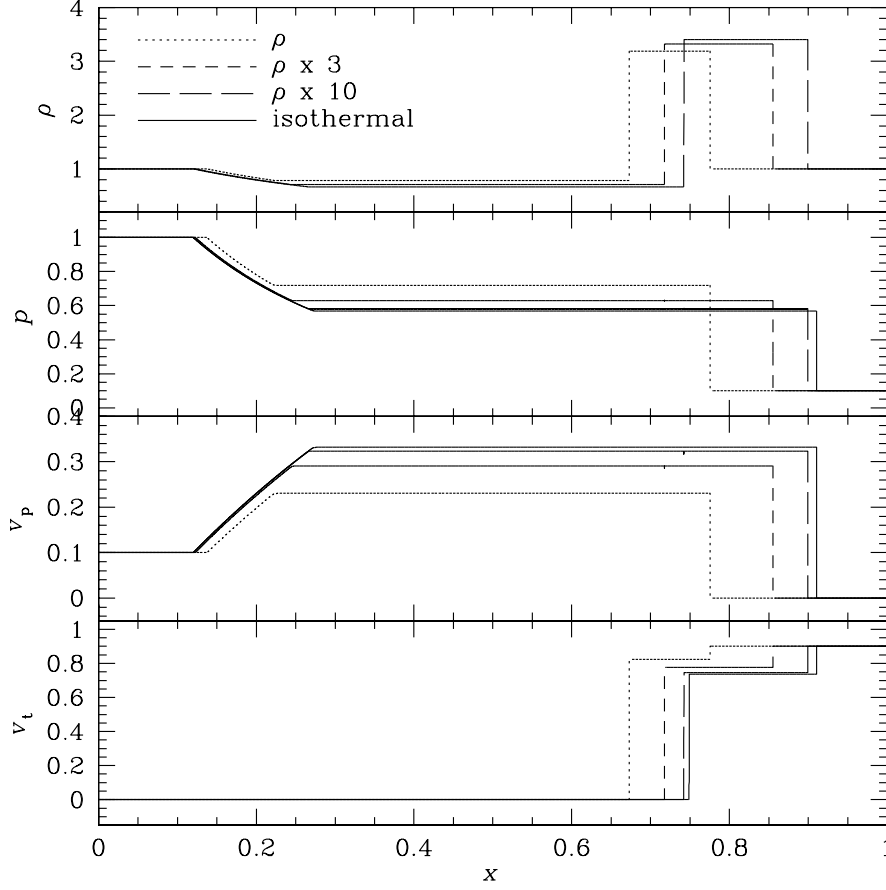


Figure 3. A relativistic shock tube from the code for isothermal flows and from the code for adiabatic flows employing RC with different initial densities,  $\rho = 1, 1/3, 1/10$ .

shock because  $\Theta \sim 1$  there. The ID solution is clearly different from the RC solution, indicating the importance of using correct EoSs.

#### 4. Code for Isothermal Flows

As in non-relativistic hydrodynamics (see, e.g., Kim et al. 1999), a code for isothermal flows, where the EoS is given by  $p = c_s^2 e$  with a constant sound speed  $c_s$ , can be built in RHDs. Such EoS arises in several important situations. Most of all, when the constituent particles are ultra-relativistic or the fluid is dominated by radiation, that is,  $e \gg \rho$ , the sound speed goes  $c_s \rightarrow 1/\sqrt{3}$  and the EoS becomes  $p = (1/3)e$ . Also the EoS in degenerate matters, such as in white dwarfs and neutron stars, may be modeled as  $p = c_s^2 e$  (see, e.g. Weinberg 1972).

Equations (2) – (3) with  $p = c_s^2 e$  form a complete, hyperbolic set of conservation equations for isothermal flows. The eigenvalues have the same form as those for adiabatic flows in (9) – (11), except two  $v_s$  instead of three; this is expected by considering the nature of weak solutions of initial value problems in RHDs (see, e.g. Smoller & Temple

1993). The left and right eigenvectors are substantially simpler than those for adiabatic flows. (They will be published elsewhere due to the page limit of this proceeding paper.) The calculation of the primitive quantities from the conserved quantities is also substantially simpler;

$$v = \frac{-\sqrt{(1 + c_s^2)^2 E^2 - 4c_s^2 M^2} + (1 + c_s^2)E}{2c_s^2 M}, \quad p = \frac{M}{v} - E. \quad (13)$$

Each component of velocity can be calculated with  $v_i = (M_i/M)v$ . Again a RHD code for isothermal flows was built based on the TVD scheme. Figure 3 shows a shock tube test comparing the results from the code for isothermal flows and the code for adiabatic flows with RC: initially  $p_L = 1$ ,  $p_R = 0.1$ ,  $v_{p,L} = 0.1$ ,  $v_{p,R} = 0$ ,  $v_{t,L} = 0$ ,  $v_{t,R} = 0.9$ , and  $t_{\text{end}} = 0.75$ . For the code for adiabatic flows,  $\rho_L = \rho_R = 1$ ,  $1/3$ , and  $1/10$ . The adiabatic solution approaches the isothermal solution, as  $\rho \rightarrow \text{small}$  or  $\Theta \rightarrow \text{large}$ . Our test indicates that for  $\Theta \gtrsim \text{a few } \times 10$ , the two solutions for isothermal and adiabatic flows become indistinguishable.

In principle, isothermal flows can be simulated with codes for adiabatic flows, as well. However, there are advantages of using codes for isothermal flows: 1) Codes for isothermal flows are faster than those for adiabatic flows, because the eigenvectors are simpler. With our codes based on the TVD scheme, the isothermal version is about 1.5 to 2 times faster than the adiabatic version. 2) Codes for isothermal flows should be numerically more robust than those for adiabatic flows, because one mode, which is the entropy mode, is less. Our tests have shown that shocks and discontinuities of  $v_i$  are better resolved with the code for isothermal flows. In addition, we expect that the numerical dissipation would be smaller in the code for isothermal flows (not shown here).

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